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EXISTENCE AND UNIQUENESS THEOREMS RELATING TO THE NAVIER-STOKES EQUATIONS FOR INCOMPRESSIBLE FLUIDS*

by

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September
1952

*Work done at Harvard University under Contract N5ori-07634, with the Office of Naval Research.

EDITORIAL NOTE

The attached report by Dr. Barrar attempts to give a complete summary of existence and uniqueness theorems for the Navier-Stokes equations for incompressible fluids and related equations. This parallels Dorothy L. Bernstein's useful summary of existence and uniqueness theorems for elliptic, hyperbolic, and parabolic equations in her book <u>Existence Theorems in Partial Differential Equations</u>. Only about ten per cent of the papers covered herein are listed in Miss Bernstein's summary.

As only about one-tenth of the references are listed in Miss Bernstein's summary, and as no other comparable compilation seems available, I hope Dr. Barrar's report will prove useful.

At the same time, as it has not been carefully edited or checked independently, persons doing original research on existence and uniqueness theorems are advised to consult original sources.

Garrett Birkhoff

I. NAVIER-STOKES EQUATIONS

A. Time Dependent

The time dependent Navier-Stokes equations for an incompressible viscous fluid are[1,2,3]:

(1a) $$\frac{\partial u_i(x,t)}{\partial t} + u_k(x,t)\frac{\partial u_i(x,t)}{\partial x_k} = \nu\nabla^2 u_i(x,t) - \frac{1}{\rho}\frac{\partial p}{\partial x_i}(x,t) + g_i(x,t) \qquad [i = 1 \dots n]$$

(1b) $$\frac{\partial u_i(x,t)}{\partial x_i} = 0 .$$

In these equations, $u_i(x, t)$ is the vector velocity; $p(x, t)$ the pressure; $\vec{g}(x, t)$ the gravitational force per unit mass; and the constants ρ and ν, the density and kinematic viscosity respectively. The usual boundary conditions are $\vec{u} = 0$ on any fixed rigid surface[4,5], and in an infinite region $\vec{u}(x, t) \to \vec{U}_\infty$ at infinity.

In the special case when $\vec{g}(x, t)$ is the gradient of a potential G (e.g., when it is the vector acceleration of gravity) by setting

(2) $$P = \frac{p}{\rho} - G ,$$

the homogeneous equation

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[1] For a derivation and related references, see [25, Vol. 1, p. 96]. Here and below, $x = (x_1, x_2, x_3)$ will denote vector position, and $\vec{u} = (u_1, u_2, u_3)$, vector velocity.

[2] $x = (x_1 \dots x_n)$

[3] Repeated indices imply summation.

[4] See [25, Vol. 2, p. 676].

[5] Only the fixed boundary problem will be discussed in this paper. For a discussion of the free boundary problem, see Lewy [42]. Also only the incompressible case will be treated, as no existence or uniqueness theorems are known for the Navier-Stokes equations for compressible fluids, with non-zero viscosity.

(3a) $$\frac{\partial u_i(x,t)}{\partial t} + u_k(x,t)\,\frac{u_i(x,t)}{\partial x_k} = \nu\nabla^2 u_i(x,t) - \frac{\partial p}{\partial x_i} \quad [i = 1\ldots n]$$

(3b) $$\frac{\partial u_i(x,t)}{\partial x_i} = 0$$

is obtained.

B. Two-Dimensional Case

In the two-dimensional case $\{$i.e., $x = (x_1 x_2)]$ if $\vec{\xi} =$ curl $\vec{u}$, only $\zeta = \xi_3$ does not vanish. Hence, by taking the curl of equation (3a) yields, in this case,

(4) $$\frac{\partial \zeta}{\partial t}(x,t) + u_i(x,t)\,\frac{\partial \zeta}{\partial x_i}\,(x,t) = \nu\nabla^2 \quad (x,t), \qquad [i = 1,2]$$

whereby the pressure potential P is eliminated.

Also in the two-dimensional case, equation (3b) permits the introduction of a locally single-valued stream function[6] $\psi(x,t)$ such that

(5) $$u_1 = \frac{\partial \psi}{\partial y}, \qquad u_2 = -\frac{\partial \psi}{\partial x}.$$

The use of ψ eliminates the necessity for (3b).

Then[7]

(6) $$\zeta = -\nabla^2 \psi,$$

and hence equation (4) may be written[8]

(7) $$\frac{\partial \nabla^2 \psi(x,t)}{\partial t} - \frac{\partial \psi(x,t)}{\partial x}\,\frac{\partial}{\partial y}\nabla^2\psi(x,t) + \frac{\partial \psi(x,t)}{\partial y}\,\frac{\partial}{\partial x}\nabla^2\psi(x,t)$$

$$= \nu\nabla^4 \psi(x,\ t)\ .$$

[6] See [52, Chap. 12, §2, equation (1)].

[7] See [52, Chap. 12, §2, equation (2)].

[8] See [52, Chap. 12, §2, equation (3)].

Similar simplifications are possible in the axially symmetric case, but we shall not describe them.

C. Steady Flow

A motion is called steady (permanent, stationary) if all the partial derivatives with respect to time vanish. In steady motions, the Navier-Stokes equations (1a)-(1b) reduce to

$$\text{(8a)}\quad \nu\nabla^2 u_i(x) = u_k(x)\,\frac{\partial u_i(x)}{\partial x_k} + \frac{\partial p(x)}{\partial x_i} + g_i(x) \qquad [i = 1 \ldots n]$$

$$\text{(8b)}\qquad \frac{\partial u_i(x)}{\partial x_i} = 0 \;.$$

The usual boundary conditions are that $\vec{u}$ is given (usually zero) as $\vec{u}^S(x, t)$ on a rigid surface S, and, in the case of an infinite region, $\vec{u}$ approaches a given $\vec{u}_\infty$ at infinity.

For steady flows, equation (7) reduces to

$$\text{(9)}\qquad \nu\nabla^4\psi(x) = \frac{\partial\psi(x)}{\partial x_2}\,\frac{\partial}{\partial x_1}\,\nabla^2\psi(x) - \frac{\partial\psi(x)}{\partial x_1}\,\frac{\partial}{\partial x_2}\nabla^2\psi(x) \;.$$

It might be conjectured that, as $t \to \infty$, the solution of any time dependent problem approaches the corresponding stationary solution. Experimental evidence supports this conjecture for equations (3) and (8) if R is sufficiently small[9], but not[10] for large R.

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[9] R is the Reynolds number, defined as the dimensionless ratio $R = \frac{vL}{\nu}$, where v is a typical velocity and L, a representative linear dimension.

[10] See [5, p. 28].

II. PROBLEMS

A. Existence and Uniqueness; Time-Dependent Case

The mathematical problems concerned with the time dependent Navier-Stokes equation (1) are of the initial value type usually associated with parabolic differential equations[11]. If Q is a region in $E^n(x_1 \ldots x_n)$ space, the problem here is to find a solution $u_i(x, t)$ of equation (1) in Q and for $0 \leqq t < T$, such that $u_i(x, 0) = u_i^0(x)$ in Q; and $u_i(x, t) = u_i^S(x, t)$, where $\vec{u}^0(x)$ is given in Q, and $\vec{u}^S(x)$ is given on S, when $x \in S$, where S is the boundary of Q. Such a problem will be denoted by $P_t(Q)$.

Various technical difficulties arise depending on whether Q is bounded or not, and on the smoothness of the boundary S of Q. When Q is unbounded, a condition such as uniformity at infinity,

$$(\alpha) \quad \lim_{|x| \to \infty} \vec{u}(x, t) = \vec{u}_\infty^0 (t) \ ,$$

or finiteness of total energy,

$$(\beta) \quad \iint_Q u_i(x,t)\, u_i(x,t) dx \leqq \text{constant}$$

must also be included to insure uniqueness[12]. When $Q = E^n$, the initial values and condition (α) or (β) are all that are given. To obtain existence and uniqueness theorems, S and the boundary values must be described more explicitly than above. This will be done in the statement of the theorems.

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[11] For the special case of Boussinesq flow [36, §334a], i.e., $V = W = 0$, $U = U(z,t)$, the Navier-Stokes equation (3) actually reduces to the heat equation

$$\frac{\partial U}{\partial t}(z,t) = \nu \frac{\partial^2 U(z,t)}{\partial z^2} \ .$$

Hence it seems plausible that the problems for the time dependent Navier-Stokes equations are of the parabolic type.

[12] See Theorem 13 below.

B. Case of Steady Flows

The mathematical problems concerned with the steady state Navier-Stokes equation (8) are boundary value problems usually associated with differential equations of elliptic type[13] (though see (34) below). If Q and S are as above, the problem is to find a solution of equation (8) in Q, that takes on preassigned values on S. This will be called problem P(Q).

C. Regularity of Solutions

For Laplace's equation $\nabla^2 U = 0$ in n-dimensions, a solution U in a region Q is analytic in the interior of Q.[14] Moreover, if a portion T of the boundary is a manifold of class C^n and the boundary value ϕ assumed by U on T is of class C^n, then in any Q-neighborhood of T, U is of class C^{n-1}.[15] For the steady state problems discussed in this paper, analogous theorems have been proved in some cases and are probably true in all cases. Those that have been proved will be listed later in this paper.

For the heat equation $\nabla^2 U(x, t) - \partial U(x, t)/\partial t = 0$ in n dimensions, a solution in a region Q for $t_0 \leq t \leq t_1$ is analytic in the space variables and of class C^∞ in t for x in the interior of Q and $t_0 < t \leq t_1$.[16] Moroever, if the initial values are of class C^n in $Q_1 \in Q$, then the solution U is of class

[13] For irrotational steady flow in the special case of zero viscosity, the equation for the stream function actually reduces to Laplace's equation $\nabla^2 \psi = 0$. (This will be discussed further on p. 16.) Hence it seems plausible that the problems for the steady state Navier-Stokes equation are of the boundary value type.

[14] See [33, p. 220].

[15] See [34].

[16] See [26, p. 290].

C^n in x and of class $C^{n/2}$ in t for $x \in Q_1$, $t_0 \leq t \leq t_1$.[17] For the time-dependent problems discussed in this paper analogous theorems have been proved in some cases and are probably true in all cases. Those that have been proved will be listed later in this paper.

D. Some Unexpected Results

To illustrate some of the pitfalls that must be avoided in dealing with these problems, the following simple examples are given:

1. To obtain uniqueness in unbounded regions, the solution must remain bounded. The following example[18] shows this:

$$H(x, t) = \frac{x}{t^{3/2}} \exp(-x^2/4t)$$

is a non-vanishing solution of[19]

$$\frac{\partial^2 U}{\partial x^2} - \frac{\partial U}{\partial t} = 0 , \tag{10}$$

for $t > 0$ and

$$\lim_{t \to 0} H(x, t) = 0, \quad \text{for all } x. \tag{11}$$

Hence, this function could be added on to any solution of the linear equation (10) to spoil uniqueness properties[20].

2. Physical intuition sometimes fails in mathematical problems. Consider the equation for the temperature in a one-dimensional infinite rod with variable heat conductivity

$$\frac{\partial u}{\partial t}(x, t) = a(x, t)\frac{\partial^2 U}{\partial x^2}(x, t) + b(x, t)\frac{\partial U}{\partial x}(x, t) + c(x, t)\,U(x, t) + b(x, t) \tag{12}$$

$$a > 0 .$$

[17]See [3].

[18]See [9].

[19]See footnote 11, p. 4 for the relevance of this equation.

[20]Note that $\lim_{\substack{x \to 0 \\ t \to 0}} H(x,t)$ is not defined. This is discussed more completely in [9].

Physically, it is plausible that if the temperature is given throughout for $t = t_0$, the temperature is determined throughout for $t > t_0$. This, however, is not necessarily true.

For equation (12) to have at most one bounded solution in the half-plane $t > 0$ approaching a given continuous function for $t = 0$, some condition such as that $\int_0^x a(y, t)^{-1/2} dy$ must diverge for $\pm \infty$ must be imposed. The following example[21] shows this.

For the self-adjoint equation

$$(13) \qquad \frac{\partial}{\partial t} u(x, t) + ch^2 x \, (ch^2 x u_x)_x = 0 \quad ,$$

if $\Phi(x)$ is the Gauss normal function

$$\Phi(x) = \frac{1}{2\pi} \int_{-\infty}^{x} e^{-y^2/2} \, dy \ ,$$

then for each $t < T$,

$$(14) \qquad u(t, x) = -1 + \Phi\left(\frac{1 - th\, x}{\sqrt{2(T - t)}}\right) - \Phi\left(\frac{-1 + th\, x}{\sqrt{2(T - t)}}\right)$$

is a bounded solution of (13) that (because $|th\, x| \leq 1$ and $\Phi(\infty) = 1$) converges to zero for all x as $t \to T-$.

3. The two-dimensional steady state Navier-Stokes equation (9) for the problem $\vec{U} = (0, 0)$ on $y = 0$, $\vec{U} = (0, 0)$ on $y = 1$ has the unique solution $\vec{U} = 0$. However, with the same boundary conditions the incompressible ideal fluid equations have an infinite number of solutions. (Any stream function $\psi(y)$ of class C^2 such that $\psi'(0) = \psi'(1) = 0$ and $\psi''(y) = F(\psi(y))$ for arbitrary F will be a solution[22].)

[21] See [21, p. 125].
[22] See [52, Chap. 11, §1, No. 2].

III. CATALOG OF THEOREMS

A Navier-Stokes Equations

Leray [37] proved the existence, but not the uniqueness of solutions to problems $P(Q^2)$, $P(Q^3)$, $P(E^2 - Q^2)$, and $P(E^3 - Q^3)$ for Navier-Stokes equation (8).[23] He remarked that his method does not generalize to $P(Q^n)$ and $P(E^n - Q^n)$.[24] More specifically, his results in E^3 are

Theorem 1. Let Q^3 be a bounded region in E^3 of class $B - h$.[25] Let $\vec{\alpha}$ be of class C^2 on S with

$$\int_S \vec{\alpha} \cdot \vec{n} \, dS = 0 .$$

Then there exists a regular solution u_i of problem $P(Q^3)$ for equation (8) with $u_i = \alpha_i$ on S.

Theorem 2: Let Q^3 and $\vec{\alpha}$ be defined as in Theorem 1. Let $\vec{a}$ be a vector defined in E^3 of divergence zero. Then there exists a regular solution u_i of the homogeneous equation (8) with $u_i = \alpha_i$ on S and $\lim_{|x| \to \infty} u_i = a_i$. Moreover,

$$J = \iiint_{E^3 - Q} \left(\frac{\partial U_i}{\partial x_k} - \frac{\partial a_i}{\partial x_k}\right)\left(\frac{\partial u_i}{\partial x_k} - \frac{\partial a_i}{\partial x_k}\right) dx_1 \, dx_2 \, dx_3$$

is bounded.

To prove theorems 1 and 2, Leray used an early form of the Leray-Schauder theorem[26]. Similar results hold in E^2.

In the time dependent case for the $P_t(E^2)$ problem, Leray [37] proved

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[23] Q^n will denote a bounded set in E^n.

[24] Technically, this is because $\int_0^1 \left(\frac{1}{r^{n-2}}\right)^2 dV_n$ does not exist for $n > 3$.

[25] Let a neighborhood of every point $p \in S$ be represented parametrically by $x_i = x_i(u,v)$, $i=1,2,3$, with all $D_1 x_i$ being h-Lipschitz continuous, i.e., $|D_1 x_i(u_1 v_1) - D_1 x_i(u_2 v_2)| \leq K[|u_1-u_2|^h + |v_1-v_2|^h]$ and the Jacobian $\partial(x_1 x_2 x_3)/\partial(uv)$ being of rank two. Then Q is called a region of class A-h. If Q is a region of class A-h, and all $D_2 x_i$ exist and are h-Lipschitz continuous, then Q is said to be of class $B - h$. This is the notation used in [43].

[26] For a statement and proof of the theorem, see [41].

Theorem 3: Let $\bar{u}_i(x_1x_2)$ and $\partial\bar{u}_i(x_1x_2)/\partial x_j$ be bounded, continuous, and belong to $L^2(E^2)$.[27] Let $\partial u_i(x_1, x_2)/\partial x_i = 0$. Then there exists a unique solution $u_i(x_1x_2t)$ of problem $P_t(E^2)$ for equation (3) for all time, with $u_i(x_1x_20) = \bar{u}_i(x_1x_2)$ and such that $u_i(x_1x_2t)$ and $\partial u_i(x_1x_2t)/\partial x_j$ belong to $L^2(E^2)$ and are bounded.

The method of proof is that of successive approximations. Starting with an existence theorem for the Stokes equation (50), Leray constructed the sequence

$$(15a)\qquad \frac{\partial u_i^{\,n}(x,t)}{\partial t} = \nu\nabla^2 u_i^{\,n}(x,t) + \frac{\partial p^n}{\partial x_i}(x,t) + g_i^{\,n}(x,t) \quad [i = 1,2]$$

$$(15b)\qquad \frac{\partial u_i^{\,n}(x,t)}{\partial x_i} = 0\ ,$$

with $g_i^{\,n} = u_k^{\,n-1}(x,t)\,\dfrac{\partial u_i^{\,n-1}(x,\ t)}{\partial x_k}$.

He then showed $u_i^{\,n}$ approached the desired solution.

For the $P_t(E^3)$ problem, Leray [39] was not able to prove the analogue of Theorem 3 for all time, but only for a neighborhood of $t = 0$. Similarly for the $P_t(Q^2)$ problem, when Q^2 is convex and of class B - h, Leray [38] was able to prove only a local existence theorem in time, even for $u_i = 0$ on S for $t \geq 0$. Dolidze [10, 14] obtained local existence theorems for $P_t(Q^2)$, $P_t(Q_3)$, $P_t(E^2 - Q^2)$ and $P_t(E^3 - Q^3)$; and Oseen [56, p. 72] preceded Leray in obtaining a local existence theorem for problem $P_t(E^3)$.

Actually, Leray [38, 39] went further than merely proving local existence theorems by successive approximations. On the one hand, he gave criteria under which his local solutions could be ex-

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27 i.e., $\iint_{E^2} u_i(x_1x_2)u_i(x_1x_2)dx_1\,dx_2 < \infty$.

tended for all time. When, for example [63] in problem $P_t(E^3)$ the given initial conditions $u_i(x, 0)$ have an axis of symmetry and when

$$\liminf_{q_0 \to \infty} q_0 \iiint_{q_0 < q} u_i(x, 0)\, u_i(x, 0)\, dV = 0 ,$$

where q is the distance from any point x to the axis of symmetry, it follows from Leray's results that the Navier-Stokes equations do have a unique solution for all time.

On the other hand, although Leray did not derive the existence of a regular solution for all time for the problem $P_t(E^3)$, he did derive the existence (but not the uniqueness) of an irregular solution[28], which will now be defined.

Definition: $u_i(x, t)$ is an irregular solution of equation (3) when it satisfies the following three conditions:

A) O does not differ from the time axis by more than a set of measure zero; where O is described as follows. Define an interval of regularity $\Theta_e T_e$ on the time axis, one in the interior of which the vector $u_i(x, t)$ is a regular solution of equation (3), and one for which this statement cannot be made for any interval containing $\Theta_e T_e$. O is the union of these disjoint intervals.

B) The function $\iiint_{E^3} u_i(x, t) u_i(x, t) dx_1 dx_2 dx_3$ is decreasing on O and the initial time $t = 0$.

C) When $t \to t'$, then $u_i(x, t') \to u_i(x, t)$ weakly in L^2 norm.

Leray [38] obtained similar results for the $P_t(Q^2)$ problem in a convex region.

[28] Leray called them turbulent solutions. Duhem [18, 1st Ser. Part 2, Chap. 3, §9] speaks of irregular solutions. Oseen [56, p. 72] is also concerned with them.

Leray [39] conjectured that some problems may have irregular but not regular solutions. He suggested that if one could find a function $v_i(x)$ belonging to $L^2(E^3)$ and satisfying

(16a) $$\nu\nabla^2 v_i - \alpha\zeta[v_i + x_k \frac{\partial v_i}{\partial x_k}] - \frac{\partial p}{\partial x_i} = \zeta\, v_k \frac{\partial v_i}{\partial x_k} \quad [\alpha > 0]$$

(16b) $$\frac{\partial v_k}{\partial x_k} = 0 \quad ,$$

then the initial values

(17) $$\bar{u}_i(x,\ 0) = \frac{v_i(2\alpha T^{-1/2}x)}{\sqrt{2\alpha T}}$$

would have no corresponding regular solution for all time, but would have the irregular solution

(18a) $$u_i = \frac{v_i[2\alpha(T - t)^{-1/2}x]}{\sqrt{2\alpha(T - t)}}, \quad \text{for} \quad t \in [0,\ T)$$

(18b) $$u_i = 0, \quad \text{for} \quad t \geq T.$$

However, as yet no one has constructed the desired vector $v_i(x)$.

Hopf [30] developed a method for establishing "weak" solutions of equations (3). Although he obtained his results much more simply than Leray, and they apply to any region and any dimension, his regularity results for "weak" solutions are not as strong as Leray's for irregular solutions. Hopf's definition of a weak solution is given below.

Definition: Let G be any open set in E^{n+1} $(x_1 \ldots x_n,\ t)$ space. Let N be the class of vectors $\vec{a}$ of class C^2 with compact support in G. If some vector $\vec{u}$ belongs to $L^2(G)$ and satisfies

$$\text{(A)} \quad \iint_G \frac{\partial a_i}{\partial t} u_i \, dx\, dt + \iint_G \frac{\partial a_i}{\partial x_\alpha} u_i u_\alpha \, dx\, dt + \nu \iint_G \frac{\partial^2 a_i}{\partial x_\beta \partial x_\beta} u_i \, dx\, dt = 0$$

for all $\vec{a}$ belonging to N, with div $\vec{a} = 0$,

$$\text{(B)} \quad \iint_G \frac{\partial h}{\partial x_i} u_i \, dx\, dt = 0 ,$$

for all h belonging to N,

then $\vec{u}$ is called a weak solution of (3).

Hopf showed that a weak solution of (3) in G which belonged to $C^2(G)$ is actually a regular solution.

In connection with (A) and (B) above, it is interesting to note that Weyl [74] showed that if U belonged to $L^2(G)$ and $\iint_G U \Delta h \, dV = 0$ for all h belonging to N, then $u = u^*$ a.e., where u^* is a regular solution of Laplace's equation. In other words, he showed that a weak solution in this case is actually a regular solution. However, the corresponding result has not been proved for equation (3), whose non-linearity makes this a difficult problem.

Another approach to $P_t(E^2)$ for equation (3) is that taken by Kampé de Fériet in [32]. He showed that the Fourier transform

$$(19) \quad z(w_1 w_2, t) = \frac{1}{4\pi^2} \iint_{E^2} \zeta(x,y,t) \exp[-i(w_1 x + w_2 y)] dx\, dy$$

of the vorticity $\zeta(x,y,t)$ associated with the two-dimensional flow of an incompressible fluid extending over the entire (x,y)-plane, with finite kinetic energy and $\zeta \in L$, transforms equation (4) into [29]

$$(20) \quad \frac{\partial}{\partial t} z(w_1,w_2,t) = -\nu(w_1^2 + w_2^2) z(w_1 w_2 t) + 2\iint_{E^2} \left(\frac{\theta_1 w_2 - \theta_2 w_1}{\theta_1^2 + \theta_2^2}\right) z(\theta_1 \theta_2 t) \bar{z}(\theta_1 + w_1, \theta_2 + w_2, t) d\theta_1 d\theta_2$$

[29] Assuming there is a solution to equation (20) leads to the problem of determining a vector $\vec{v}$ in a given region Q, when curl $\vec{v}$ and div $\vec{v}$ are given in Q, and $\vec{v}_n$ is given on S. A proof of uniqueness can be found in [52, p. 457] and a proof of existence in [70, pp. 16-20].

Bellman [4], starting with equation (20) and the initial condition $z(w_1w_2,0) = \phi(w_1w_2)$, derived by successive approximations

Theorem 4: If $\max |\phi(w_1w_2)|$ is sufficiently small, there is a solution of (20) which is unique $z(w_1w_2 0) = \phi(w_1w_2)$ and satisfies the inequality[30]

$$|z(w_1w_2t)| \leq \frac{8 \max |\phi(w_1w_2)|}{[1 + \nu(w_1^2 + w_2^2)t]^2} .$$

B. Incompressible, Non-Viscous Fluids[31]

In the case when $R = \infty$, equation (1) reduces to the equations of motion of an ideal incompressible fluid[32]

$$\text{(21a)} \quad \frac{\partial u_i(x,t)}{\partial t} + u_k(x,t)\frac{\partial u_i(x,t)}{\partial x_k} = \frac{\partial P(x,t)}{\partial x_i} + g_i(x,t) \quad [i=1..n]$$

$$\text{(21b)} \quad \frac{\partial u_i(x,t)}{\partial x_i} = 0 .$$

The usual boundary conditions are $\vec{u} \cdot \vec{n} = 0$, instead of $u = 0$, on any fixed rigid surface.

In this case, equation (4) reduces to

$$\text{(22)} \quad \frac{\partial \zeta(x,t)}{\partial t} + u_i(x,t)\frac{\partial \zeta_i(x,t)}{\partial x_i} = 0 \qquad [i = 1,2] ,$$

which means that ζ is constant along the path of every particle[33].

When $R = \infty$, equation (8) reduces to

$$\text{(23a)} \quad u_k(x)\frac{\partial u_i(x)}{\partial x_k} = \frac{\partial P(x)}{\partial x_i} + g_i(x) \qquad [i = 1 \ldots n]$$

$$\text{(23b)} \quad \frac{\partial u_i(x)}{\partial x_i} = 0 .$$

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[30] Although this is no improvement over Leray's result, the approach is interesting.

[31] For ideal flow for compressible fluids, see [50, 69, and 8].

[32] See [52, Chap. 10, §1, No. 6].

[33] [ibid., No. 1].

f) The divergence is zero

g) The normal component of the velocity at the boundary is zero

h) The paths of the moving point $dx/dt = u$, $dy/dt = v$, with $x = a$, $y = b$, at $t = 0$, exist; and at each instant there is a one to one, continuous area preserving relation between (a, b) and (x, y), $(a, b) \in \bar{R}$, $(x, y) \in \bar{R}$

i) The vortex density is constant along the moving point, and in each finite time interval satisfies a uniform Lipschitz condition with respect to x, y, and t.

Lichtenstein [46] earlier proved the following local type of existence theorem for the plane.

<u>Theorem 6</u>: Let a solution of problem $P_t(E^2)$ of the ideal incompressible fluid equations exist for $t_0 \leq t \leq t_1$ with initial values $\tilde{U}_0(x)$. Then if the initial values $\tilde{U}_0(x)$ are changed slightly[35] to $\tilde{\tilde{U}}_0(x)$, a unique solution to problem $P_t(E^2)$ will exist for the initial conditions $\tilde{\tilde{U}}_0(x)$ for $t_0 \leq t \leq t_1$.

Although this theorem does not constitute an improvement over the theorems of Wolbiner, Schaeffer and Hölder, Maruhn [49] extended it to axially symmetric initial conditions in E^3 which is not covered by their two-dimensional theorems.

In the steady state case, restricted to irrotational motion, the problem is reduced to the Neumann problem in potential theory[36]. The existence and uniqueness theorems for this problem can be found in Kellogg [33] and Gunther [27].

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[35] $|\tilde{U}_i(x) - \tilde{\tilde{U}}^o(x)| < \epsilon \quad |D_i \tilde{U}^o_j(x) - D_i \tilde{\tilde{U}}^o_j(x)| < \epsilon \quad [i = 1,2]$

[36] See [52, Chap. 11, §3, p. 440].

f) The divergence is zero

g) The normal component of the velocity at the boundary is zero

h) The paths of the moving point $dx/dt = u$, $dy/dt = v$, with $x = a$, $y = b$, at $t = 0$, exist; and at each instant there is a one to one, continuous area preserving relation between (a, b) and (x, y), $(a, b) \in \bar{R}$, $(x, y) \in \bar{R}$

i) The vortex density is constant along the moving point, and in each finite time interval satisfies a uniform Lipschitz condition with respect to x, y, and t.

Lichtenstein [46] earlier proved the following local type of existence theorem for the plane.

Theorem 6: Let a solution of problem $P_t(E^2)$ of the ideal incompressible fluid equations exist for $t_0 \leq t \leq t_1$ with initial values $\tilde{U}_0(x)$. Then if the initial values $\tilde{U}_0(x)$ are changed slightly[35] to $\approx{U}_0(x)$, a unique solution to problem $P_t(E^2)$ will exist for the initial conditions $\approx{U}_0(x)$ for $t_0 \leq t \leq t_1$.

Although this theorem does not constitute an improvement over the theorems of Wolbiner, Schaeffer and Hölder, Maruhn [49] extended it to axially symmetric initial conditions in E^3 which is not covered by their two-dimensional theorems.

In the steady state case, restricted to irrotational motion, the problem is reduced to the Neumann problem in potential theory[36]. The existence and uniqueness theorems for this problem can be found in Kellogg [33] and Gunther [27].

35 $|\tilde{U}_i(x) - \approx{U}^o(x)| < \epsilon \quad |D_i \tilde{U}_j^o(x) - D_i \approx{U}_j^o(x)| < \epsilon \quad [i = 1,2]$

36 See [52, Chap. 11, §3, p. 440].

C. Burgers' One-Dimensional Analogue

To gain better insight into the Navier-Stokes equations, Burgers [6 and 7] and Hopf [29] have studied the one-dimensional analogue

$$(24) \qquad \frac{\partial u(x_1,t)}{\partial t} + u(x_1,\ t)\,\frac{\partial u(x_1,t)}{\partial x_1} = \nu\,\frac{\partial^2 u(x_1,t)}{\partial x_1{}^2} \ .$$

This is evidently a non-linear parabolic equation.

Hopf [29] observed that the transformation

$$(25) \qquad \phi = \exp \quad - \frac{1}{2\nu} \int u\, dx$$

or

$$(26) \qquad u = -\,2\,\nu\phi_x/\phi$$

takes a solution of Burgers' equation (24) into the heat equation

$$(27) \qquad \phi_t = \nu\phi_{xx} \ ,$$

and vice versa. (See also J. D. Cole, Quart. Appl. Math. 9 (1951) 225-236.)

Thus by using known theorems for (27), Hopf established

Theorem 7: Suppose that $u_o(x)$ is integrable in every finite x-interval and that

$$(28) \qquad \int_o^x u_o(\xi)d\xi = o(x^2), \quad \text{for} \quad |x| \text{ large} .$$

Then

$$(29) \qquad u(x,t) = \frac{\int_{-\infty}^{\infty} \frac{x-y}{t} \exp \quad - \frac{1}{2\nu} F(x,\ y,\ t) \quad dy}{\int_{-\infty}^{\infty} \exp \quad - \frac{1}{2\nu} F(x,\ y,\ t) \quad dy} \ ,$$

where

$$(30) \qquad F(x,\ y,\ t) = \frac{(x-y)^2}{2t} + \int_o^y u_o(\eta)\, d\eta$$

is a regular solution of (24) in the half-plane $t > 0$ that satisfies the initial conditions

$$(31) \qquad \int_0^x u(\xi, t)\, d\xi \to \int_0^a u_0(\xi)\, d\xi, \quad \text{as } \begin{cases} x \to a \\ t \to 0 \end{cases},$$

for every a. If, in addition, $u_0(x)$ is continuous for $x = a$, then

$$(32) \qquad u(x, t) \to u_0(a), \quad \text{as } x \to a, \; t \to 0.$$

A solution of (24) which is regular in some strip $0 < t < T$ and satisfies (31) for each value of the number a, necessarily coincides with (29) in the strip.

<u>The Case $R \to \infty$</u>. If $u_1(x, t, \nu)$ is a solution of the Navier-Stokes equation (3) for a given ν in a given region Q, taking prescribed values on S, the boundary of Q, a problem is to find what limit (if any) $u_1(x, t, \nu)$ approaches as $R \to \infty$.

From results obtained for equation (24), Hopf [29, p. 201] conjectured that in the interior of Q as $R \to \infty$, $u_1(x, t, \nu)$ approaches a "generalized" solution of the ideal incompressible equations (21).[37,38] The approach, however, cannot be uniform at the boundary because a viscous fluid adheres to the wall, and an ideal fluid slides along it[39].

D. The Boundary Layer Equation

To study the behavior of a fluid in the neighborhood of the boundary as $R \to \infty$, Prandtl [62] derived the <u>boundary layer equation</u>[40,41]

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[37] This "generalized" solution is a weak solution of (21) in the sense given on p. 11 of this report.

[38] Wasow [71] treats the case $\lim_{\lambda \to \infty} \nabla^2 u(x,y) + \lambda \frac{\partial u}{\partial x} = \lambda f(x,y)$. Oseen [56, Part 3] also treats the linear case.

[39] See [73, p. 383].

[40] $x = \mathcal{L}\xi$, $y = \delta\eta$ where $\delta/\mathcal{L} \sim 1/\sqrt{R}$.

[41] For physical background, see [25, Vol. 1, Chap. 2] and [17].

(33) $$\frac{\partial \psi}{\partial \eta} \frac{\partial^2 \psi}{\partial \xi \partial \eta} - \frac{\partial \psi}{\partial \xi} \frac{\partial^2 \psi}{\partial \eta^2} - \frac{\partial^2 \psi}{\partial \eta \partial t} - \frac{\partial^3 \psi}{\partial \eta^3} = f(\xi) \quad ,$$

when η is the straight-line boundary of the two-dimensional region Q, and $f(\xi)$ is given.

The boundary layer equation in the steady state case may be conveniently given in a form due to von Mises [51]. Setting $z = k^2 - (\frac{\partial \psi}{\partial \eta})^2$ as the dependent variable, and ψ and ξ as independent variables, von Mises reduced (33) to the non-linear parabolic equation

(34) $$\frac{\partial z}{\partial \xi} = \sqrt{k^2 - z} \; \frac{\partial^2 z}{\partial \psi^2} + h(\xi) z \; .$$

This form is very convenient for theoretical discussion.

For the von Mises equation (34), Piscounov [61] proved

<u>Theorem 8</u>: Let $z(0, \psi) = \phi(\psi)$, $z(\xi, 0) = k(\xi)$ and $z(\xi, \infty) = 0$. Moreover, let

a) $\phi(0) = k(0)$, $\phi(\infty) = 0$

b) $k(0) \geq \phi(\psi)$ for $\psi > 0$

c) $\phi(\psi) \leq k(0) - c\psi$ for some $\epsilon_1 > 0$, with $\psi < \epsilon_1$

d) $k(\xi)$ be continuous and non-decreasing[42].

Then under these boundary conditions, the equation

(34') $$z_{\psi\psi} = (K(\xi) - z)^{-1/2} \, z_\xi$$

has a solution. Moreover, let

e) $\phi''(\psi) \geq 0$ for some $\epsilon_2 > 0$ with $\psi \leq \epsilon_2$.

Then the solution is unique.

When the boundary is an angle of $\pi\lambda$, Falkner and Skan [19]

[42] This means a favorable pressure gradient, i.e., the flow is accelerating.

reduced the boundary layer equation to an ordinary differential equation. Weyl [73] gave an existence but not a uniqueness theorem for the differential equation. Using conformal coordinates (ξ, η) with $ds^2 = dx_1{}^2 + dx_2{}^2 = (d\xi^2 + d\eta^2)\, e(\xi, \eta)$, he obtained the more general form

$$(35) \qquad h(\xi)[k^2(\xi) - (\frac{\partial \psi}{\partial \eta})^2] + \frac{\partial^2 \psi}{\partial \xi \partial \eta}\frac{\partial \psi}{\partial \eta} - \frac{\partial^2 \psi}{\partial \eta^2}\frac{\partial \psi}{\partial \xi} = \frac{\partial^3 \psi}{\partial \eta^3},$$

where a) $\eta = 0$ is the boundary, b) $h(\xi) = \frac{1}{2}\, d \log e(\xi, 0)/d\xi$, c) $\partial\psi/\partial\eta = u_1(\xi, \eta) \to k(\xi)$ for $\eta \to \infty$.

E. <u>General Theory of Non-Linear Parabolic Equations</u>.

The only results for non-linear parabolic equations apply to non-singular equations for bounded regions. Although these results cannot be directly applied to the von Mises equation, it is conceivable that through some suitable limiting process they can.

Gevrey [23, §§28-29], through successive approximations, obtained what are probably the best results for non-linear parabolic equations. To indicate the scope of his results, they are given below in a simplified form. By coordinate transformations and extension of boundary values to all of (x,t)-space, the theorems can be applied under more general conditions than given here.

Let $T = \{x, y \mid x \in [0, 1],\ y \in [0, 1]\}$.

<u>Theorem 9</u>: Let $f(x, y, z, p)$ [43] be such that

$$(36) \qquad |f(x_1y_1z_1p_1) - f(x_2y_2z_2p_2)| \leq K[|x_1-x_2|^\gamma + |y_1-y_2|^\gamma + |z_1 + z_2| + |p_1 - p_2|]$$

for $(x, y) \in T$, $|z|, |p| < N$. Then there exists a unique solution to the problem[44]

[43] $p = \partial z/\partial x$, $q = \partial z/\partial y$.

[44] Albers [1] generalized this to n dimensions.

(37) $$\frac{\partial^2 z}{\partial x^2} - \frac{\partial z}{\partial y} = f(x,y,z,p) \text{ in } T \text{ with } z = 0 \text{ on } \begin{cases} x = 0 \\ x = 1 \\ y = 0 \end{cases}.$$

Theorem 10: Let $g(x, y, z, p, q)$ be such that $\partial g/\partial y$, $\partial g/\partial z$, $\partial g/\partial p$, $\partial g/\partial q$ exist and satisfy a condition similar to (36) with the addition of a term in $|q_1 - q_2|$. Let $\partial g/\partial q > 0$. Then there exists a unique solution to the differential equation

(38) $$\frac{\partial^2 z}{\partial x^2} - \frac{\partial z}{\partial y} = g(x,y,z,p,q) \text{ in } T, \text{ with } z = 0 \text{ on } \begin{cases} x = 0 \\ x = 1 \\ y = 0 \end{cases}.$$

Another approach to the existence theorems for the von Mises and other non-linear parabolic equations is through difference equations. Luckert [48] assumed equation (35') had a solution, and showed how to obtain it by difference equations. John [31], among others, proved the convergence of solutions of difference equations to solutions of parabolic equations in many cases[45].

F. Oseen Equation.

In a given unbounded region Q, equation (1) simplifies to the linear Oseen equation[46] by replacing the term $u_k \frac{\partial u_i}{\partial x_k}$ by the approximation $\bar{u}_k \frac{\partial u_i}{\partial x_k}$, where $\bar{u}_k = \lim_{|x| \to \infty} u_k(x,t)$. This gives an asymptotic approximation for equation (1) at large distances. By rotating the x_1 axis parallel to the $\bar{u}_k$ vector, the Oseen equation reduces to

(39) $$\frac{\partial u_i(x,t)}{\partial t} + \bar{u}_1 \frac{\partial u_i(x,t)}{\partial x_1} = \nu \nabla^2 u_i(x,t) + \frac{\partial p(x,t)}{\partial x_i} + g_i(x,t),$$

$$[i = 1 \ldots n];$$

as usual, we also assume (1b). In the steady case, (39) reduces to

[45] See John's [31] bibliography on this subject.

[46] See [60].

(40) $\nu\nabla^2 u_i(x) + \dfrac{\partial p(x)}{\partial x_i} - \bar{u}_1 \dfrac{\partial u_i(x)}{\partial x_1} = g_i(x), \quad [\ i = 1 \ldots n].$

The two-dimensional homogeneous equation (4) for the stream function becomes

(41) $$\frac{\partial \nabla^2\psi(x,\ t)}{\partial t} - \bar{u}_1 \frac{\partial \nabla^2\psi(x,t)}{\partial x_1} = \nu \nabla^4\psi(x,t) \ ,$$

and in the steady state case,

(42) $$\nu\nabla^4\psi(x) = - \bar{u}_1 \frac{\partial^2\psi(x)}{\partial x_1} \ .$$

Oseen [57, 58, 59] treated the homogeneous equations (39) and (40) in E^2 and E^3 but always in a domain of the form $x_2 \geq 0$, corresponding to a flow extending to infinity on one side of a fixed streamline. His method is that of integral equations. Fáxen [20], also by integral equations, derived

<u>Theorem 11</u>: Problems $P(Q^3)$ and $P(E^3 - Q^3)$ both have one and only one solution when Q^3 is of class B - h and when $u_i = \alpha_i$ on S, for continuous α_i, and when

A. For $P(Q^3)$, $\int_S \vec{\alpha} \cdot \vec{\eta}\, dS = 0$

B. For $P(E^3 - Q^3)$, $\vec{U} \to 0$ as $1/r$.

The same result applies in E^2. In both the two- and three-dimensional cases the solutions are of class C^∞ in the interior.

G. <u>Fokker-Planck Equation</u>

The Oseen equation is similar in form to the Fokker-Planck equation

(43) $$\frac{\partial u(x,t)}{\partial t} = \sum a_{ij}(x,t) \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum b_i(x,t) \frac{\partial u}{\partial x_i} + c(x,t)u + f(x,t)$$

with the matrix $||a_{ij}||$ positive definite. Thus (41) is seen as a special case of (43), if $\nabla^2\psi = \zeta$ is taken as the dependent variable. Equation (43) applies to diffusion with variable, nonisotropic diffusivity.

For the Fokker-Planck equation, Feller [21] defined a fundamental solution $u(x, t; \xi, \tau)$ for the homogeneous equation

(12') $u_t(x,t) = a(x,t)u_{xx} + b(x,t)u_x(x,t) + c(x,t)u,$

as follows:

1. As a function of x, t; $u(x,t; \xi, \tau)$ satisfies (12')

2. As a function of ξ, τ; $u(x,t; \xi, \tau)$ satisfies the adjoint equation $u_t(x,t) = (a(x,t)u_x)_x - (bu)_x + cu$

3. $$\lim_{t \to} \int_a^b u(x,t;\xi,\tau)\, f(\xi)\, d\xi = \begin{cases} f(x) & a < x < b \\ 0 & x < a \text{ or } b < x \end{cases}$$

for $f(x)$ continuous. (If the range of x is not finite, some additional condition on $f(x)$ is needed. A sufficient condition is that $f(x)$ is bounded.)

4. The above properties hold throughout the infinite range

$$-\infty < x < \infty, \qquad t_0 < \tau < t \leq t_1 ,$$

and

$$\int_{-\infty}^{\infty} u(x,t,s,\tau)u(s,t;\xi,\tau)ds = u(x,t; \xi,\tau) .$$

Under the assumptions

A'. For $t_0 \leq t \leq t_1$ and all x; $a, a_t, a_x, a_{xx}, b, b_x, c$ are γ-Lipschitz continuous in x and t.

B'. $a, 1/a, \lambda, \lambda_x$, c are bounded where $\lambda = b - a_x/2 + \sqrt{a}\ \phi_t$ and where $\phi(x,t) = \int_0^x a(y,t)^{-1/2}\, dy,$

he proved in [21] the existence of a fundamental solution for (12'). Dressel [15, 16] generalized this result to equation (43). Recently Weber [72] extended it to the equation

$$\sum_{ij} a_{ij} \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{n} (x_i \frac{\partial u}{\partial y_i} + a_i \frac{\partial u}{\partial x_i}) + aU + \frac{\partial u}{\partial t} = 0.$$

Fundamental solutions are used to prove the existence of solutions of initial value problems. Using the concept of a fundamental solution, Feller [21] showed that $P_t(E^1)$ for equation (12) has a unique bounded solution U that approaches a continuous $\phi(x)$ for $t \to 0$, when $f(x, t)$ has continuous bounded first derivatives, and the coefficients satisfy assumptions A', B' above.[47]

When the coefficients satisfy γ-Lipschitz conditions, problem $P_t(Q^n)$ for equation (43) has one and only one solution for continuous initial and boundary values as shown in Barrar [3].

In order to treat singular Fokker-Planck equations, for example, when $a(r_1) = 0$ in

(44) $$u_t(t,x) = a(x)u_{xx}(t,x) + b(x)u_x(t, x)$$

and its adjoint

(45) $$v_t(t,x) = \frac{\partial}{\partial x}\left[\frac{\partial}{\partial x}(a(x)\, v(t,x)) - b(x)\, v(t, x)\right]$$

in the interval $[r_1, r_2]$, Feller [22] used an approach which depends on the Hille-Yosida theorem for generators of semi-groups[48].

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[47] Recently John [31, p. 165] extended this theorem to apply when $\phi(x)$ is bounded and Riemann integrable.

[48] Hille-Yosida theorem: Let S be a contraction semi-group with range dense in the Banach space X in which positive elements are defined. [i.e., S is a set of transformations T_t (with $t > 0$) such that to each $x \in X$ and each $t > 0$, there corresponds an element $T_t x \in X$ such that $T_{t+s}x = T_t(T_s x)$. Furthermore,

a. $||T_{t+h}\, x - T_t x|| \to 0$ as $h \to 0$
b. $||T_t x|| \leq ||x||$
c. $x \geq 0$ implies $T_t x \geq 0$]

The infinitesimal generator A is an additive operator whose domain is dense in X and such that to each $x \in X$ and each $\lambda > 0$ there exists a unique y_λ belonging to domain of A with

A. $\lambda y_\lambda - Ay_\lambda = x$
B. $\lambda\, ||y_\lambda|| \leq ||x||$
C. $y_\lambda \geq 0$ whenever $x \geq 0$

Conversely, an additive operator with these properties is the infinitesimal generator of a contraction semi-group with range dense in X. The inverse $y_\lambda = (\lambda I - A)^{-1}$ is called the resolvent of S. For a proof, see [76].

He assumed $a'(x)$, $b(x)$ are continuous, but not necessarily bounded, in the open interval $(r_1 r_2)$ and $a > 0$. To express his results, he defined

$$w(x) = \exp \left[- \int_{x_0}^{x} b(s)\, a^{-1}(s)\, ds \right],$$

where $x_0 \in (r_1 r_2)$ is fixed. He used the regularity condition that the boundary r_j is called regular if $w(x)$ belongs to the space of integrable functions on (x_0, r_j), $\mathcal{L}(x_0, r_j)$, and $a^{-1}(x) w^{-1}(x)$ belongs to $\mathcal{L}(x_0, r_j)$. Thus, for example, for the equation $u_t = u_{xx}$, r_j is regular if and only if r_j is finite. With this notation, he derived

<u>Theorem 12</u>: Under the above conditions, when none of the boundaries is regular, there exists one and only one fundamental solution common to (44) and (45). When one or both boundaries are regular, then a necessary and sufficient condition that (44) and (45) have a common fundamental solution is that there exist constants p_j, q_j such that

$$(46) \quad q_j \lim_{x \to r_j} u(t,x) + p_j (-1)^j \lim_{x \to r_j} w^{-1}(x) u_x(t,x) = 0$$

$$(47) \quad q_j \lim_{x \to r_j} w(x) a(x) v(t,x) + p_j (-1)^j \lim_{x \to r_j} ([a(x) v(t,x)]_x - b(x)\, v(t,x)) = 0,$$

respectively at the regular boundaries.

Feller called (46) and (47) the "generalized classical boundary conditions." His method of proof works only when the coefficients in (44) and (45) do not depend on the time.

Gevrey [24] obtained existence theorems for $P_t(Q^n)$ for the parabolic system of equations

$$(48) \quad \sum_{i,j}^{n} a_{ij}^{k} \frac{\partial^2 U_k}{\partial x_i \partial x_j} + b_k \frac{\partial U_k}{\partial t} + \sum_{hi} b_{hi}^{k} \frac{\partial U_h}{\partial x_i} + \sum_{h} c_h^k U_h = f_k \; [k=1..n].$$

when a_{ij}^k is positive definite, $b_k < 0$ and when Q is of class $B-h$ and the coefficients are γ-Lipschitz continuous. However, it does not seem possible to apply these results to Oseen's equation.

At this point it is interesting to note the following

<u>Theorem 13</u>: Let (a_{ij}^k) for $k = 1 \ldots n$ of (48) be a symmetric positive definite form in Q^n for $0 \leq t \leq T$. Let $b^k < 0$. Let b_{hi}^k, a_{ij}^k, c_h^k, f_k of (48) all be γ-Lipschitz continuous in Q^n for $0 \leq t \leq T$. Let Q be of class $B-h$. Then there exists one and only one regular solution to problem $P_t(Q_n)$ for $0 \leq t \leq T$, for preassigned values of U_k;[49] if and only if the homogeneous problem[50] has only the trivial solution $U_k \equiv 0$, $k = 1 \ldots n$.

<u>Proof</u>:[51] The theorem will be proved, without loss of generality for $U_k(x,t) = 0$ for $x \in S$ and $t = 0$. Let H be the Banach space of vectors $\vec{\psi} = (\psi_1 \ldots \psi_n)$ such that each ψ_i is γ-Lipschitz continuous, with $\|\vec{\psi}\| = \sum_{i=1}^{n} \|\psi_i\|_\gamma^{Q_n}$.[52] For any given $\vec{\psi} \in H$, let $\vec{U}(\vec{\psi}) = (U_1 \ldots U_n)$ be the vector such that

$$\sum_{ij} a_{ij}^k \frac{\partial^2 U^k}{\partial x_i \partial x_j} - \frac{\partial U^k}{\partial t} = \psi_k, \quad U^k(x,t) = 0 \begin{cases} x \in S \\ t = 0 \end{cases} \qquad [k = 1 \ldots n]$$

Then let

$$L(\vec{U}(\vec{\psi}))_k = \sum_{hi} b_{hi}^k \frac{\partial U_h}{\partial x_i} + \sum c_h^k U_h$$

and

$$\vec{F} = (f_1 \ldots f_n).$$

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[49] i.e., $U_k(x,0) = U_k^0(x)$; $U_k(x,t) = U^1(x,t)$ for $x \in S$, with $U_k^0(x)$ and $U^1(x,t)$ given continuous functions thatagree when $t = 0$.

[50] i.e., $U_k^0(x) = U^1(x,t) = 0$ in footnote 49 above, and $f_k \equiv 0$, $k = 1 \ldots n$.

[51] The proof is in the spirit of Schauder [65, Chap. 4].

[52] $\|\psi_i\|_\gamma^{Q_n} = \max_{Q_n} |\psi_i| + C_\gamma$, where C_γ = g.l.b. of all C's such that $|\psi_i(x_1,t_1) - \psi_i(x_2,t_2)| \leq C[|x_1 - x_2|^\gamma + |t_1 - t_2|^\gamma]$ and where $x_1, x_2 \in Q^n$.

Using this notation, equation (48) may be written

$$\vec{\psi} + L(\vec{U}(\vec{\psi})) = F, \qquad \vec{\psi}, \vec{F} \in H \quad .$$

From results in [3], it follows that $L(\vec{U}(\vec{\psi}))$ is a completely continuous linear operator taking H into H. Thus by the Hildebrandt-Riesz theorem[53], equation (48) has a solution if and only if the homogeneous equation has only the trivial solution[54].

H. Stokes's Equations (Case $R \downarrow 0$)

Equation (1) may be plausibly simplified at low Reynolds number $R << 1$ by assuming that all velocities are very small, and thus completely neglecting the term $u_k \, \partial u_i / \partial x_k$ quadratic in the velocities. The resulting Stokes equation[55]

(49a) $$\frac{\partial u_i(x,t)}{\partial t} = \nu \nabla^2 u_i(x,t) + \frac{\partial p}{\partial x_i}(x,t) + g_i(x,t) \quad [i = 1 \ldots n]$$

(49b) $$\frac{\partial u_i(x,t)}{\partial x_i} = 0$$

or, in the steady state,

(50a) $$\nu \nabla^2 u_i(x) + \partial p / \partial x_i = g_i(x,t) \quad [i = 1 \ldots n]$$

(50b) $$\partial u_i(x) / \partial x_i = 0$$

is mathematically a special case of the Oseen equation.

The two-dimensional homogeneous equation for the stream function becomes

(51) $$\frac{\nu \nabla^2 \psi(x,t)}{\partial t} = \nu \nabla^4 \psi(x,t) \; .$$

In the steady state, it becomes the biharmonic equation

(52) $$\nabla^4 \psi = 0 \quad .$$

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[53] For a statement and proof, see [2, p. 150].

[54] No known uniqueness theorem is applicable to equation (48). The one in [24] uses integral equations.

[55] See [68].

For the Stokes equations, Odqvist [54] derived a result similar to Theorem 11 for $P(Q^3)$, $P(E^3 - Q^3)$, $P(Q^2)$ and $P(E^2 - Q^2)$ under the somewhat weaker restriction that Q is of class Ah. His results also cover the inhomogeneous equation when g_i is bounded and γ-Lipschitz continuous. Odqvist was able to characterize the behavior of the solutions of the Stokes equation more precisely than Faxén could characterize the behavior of the solutions of the Oseen equation. For example, Odqvist proved that if Q is of class Bh, and the boundary values are of class C^2, then the solution is of class C^2 in $\bar{Q}$.

Leray [37, 38] discussed $P_t(E^2)$ and $P_t(E^3)$. For example, in [39] he derived

Theorem 14: Let $\bar{u}_i(x)$ be continuous and belong to $L^2(E^3)$. Let there exist a continuous function $f(t)$ for $t \in [0, t]$ such that

$$\iiint_{E^3} g_i(x,t) g_i(x,t) dv \leq f(t) .$$

Let $\partial g_i(x,t)/\partial x_j$ exist. Then $P_t(E_3)$ for Stokes equation (49) has one and only one solution $u_i(x,t)$ with $u_i(x,0) = \bar{u}_i(x)$, and such that there exists a continuous function $g(t)$ defined for $t \in [0, T]$ such that

$$\iiint_{E^3} u_i(x,t) u_i(x,t) \leq g(t) .$$

He derived a similar result in [38] for $P_t(Q^2)$ with Q convex and of class B - h, and with $u_i = 0$ on S. In [40] he gave a method (without details) to transfer $P_t(Q^2)$ and $P_t(E^2 - Q^2)$ to Volterra integral equations, and thus gave a way to obtain existence theorems.

For $P_t(Q^3)$ and $P_t(E^3 - Q^3)$, Odqvist [55], by a Laplace transformation, orthogonal series, and integral equations, obtained the corresponding existence theorems when Q^3 is of class

B^h. However, Odqvist, due to technical difficulties, had to assume $D_k(u_i(x, 0)) = 0$, $k = 0, 1, \dots 4$ for $x \in S$.

Dolidze [11, 12, 13] was also concerned with $P_t(Q^3)$, $P_t(E^3 - Q^3)$, $P_t(Q^2)$ and $P_t(E^2 - Q^2)$. The existence theorems in [13] may possibly be more general than those in Leray [38] and Odqvist [55]. However, J. Kravtchenko [Math. Rev. 9 (1948) 116-117] has said that Dolidze's reasoning is very condensed and his hypotheses are not clear. Kneale [35] treated the axially symmetric case, deducing his existence theorems from known ones for the biharmonic equation.

Uniqueness theorems for the Stokes equations are readily proved. The proof for the steady state case is given in Lichtenstein [47, p. 394]; the proof for the time dependent case , in Leray [39, p. 215][55, 56].

For the biharmonic equation, Schröder [66] obtained existence and uniqueness theorems for $\nabla^4\psi = 0$ for $P(Q^2)$, $P(Q^3)$, $P(E^2-Q^2)$, and $P(E^3 - Q^3)$ when Q is of class Ah, with $\partial\psi/\partial x_i \dashrightarrow a_i$ on S. For the $P(E^3 - Q^3)$ existence and uniqueness theorem, he postulated that $\partial\psi/\partial x_i$ vanished as $1/r$. For the $P(E^2 - Q^2)$ existence and uniqueness theorem, he postulated that $\partial\psi/\partial x_i$ was bounded at infinity. This is connected with the Stokes paradox, which can be stated as follows.

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[55] Since the Navier-Stokes equations are not linear, a more complicated proof is needed for them. However, it is very similar to the proof above. See [39, p. 221]. Leray's proof also works for Oseen's equation.

[56] Although the proof on p. 25 does not apply to the time dependent Stokes and Oseen equations, it may be conjectured that it can be modified to do so. Since uniqueness theorems are known for the Stokes and Oseen equations, the proof would then yield an existence theorem. It would also yield stronger bounds on the behavior of the solution than previously proved, because with this proof it is possible to go directly from the Fokker-Planck to the Oseen and Stokes equations. There is a similar relationship between elliptic equations and the steady state Stokes and Oseen equations.

Although a solution of the steady state homogeneous Stokes equation (50) exists for the boundary conditions $\vec{U} = 0$ on the surface of a sphere and $\vec{U} \to \vec{U}_\infty$ at infinity, the corresponding solution for the two-dimensional problem of $\vec{U} = 0$ on the surface of an infinite circular cylinder does not exist[57].

Odqvist [54] also obtained existence theorems for the two-dimensional biharmonic equation, but his results are not as general, nor does he describe the behavior at infinity as specifically as Schröder.

In Schröder [67], properties of solutions of $\nabla^4\psi = 0$ are studied very carefully. For example, he proves that if S is a manifold of class C^n, and the boundary values are of class C^n, then ψ is of class C^n in $\bar{Q}$. Other properties for them can be found in Nicolesco [53]. In [53, §8], for example, he gives the proof that a biharmonic (in fact, a polyharmonic) function defined in Q is analytic in the interior of Q.

From known results about Poisson's equation $\nabla^2 U = g$, it is possible to construct a particular solution of $\nabla^4 U = f$.[58] Thus because of the linearity of the biharmonic equation, all existence theorems for the homogeneous equation also apply to the inhomogeneous.

Similarly, just as particular solutions of Poisson's equation in infinite space may be constructed with $1/r$, particular solutions of Oseen's and Stokes' equations may be constructed with the tensors given in [56], so that the results for the homogeneous equation again apply to the inhomogeneous.

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[57] See [5, p. 33] and [35].

[58] Let $\tilde{U} = \int_Q f \frac{1}{r} dV_3$. If all $D_n f$ are γ-Lipschitz continuous, then all $D_{n+2}\tilde{U}$ are γ-Lipschitz continuous; furthermore, $\nabla^2\tilde{U} = f$ for f γ-Lipschitz continuous [43, §30]. From these results, it can be seen that $\nabla^4\hat{U} = f$ where $\hat{U} = \int_Q \tilde{U} \frac{1}{r} dV_3$. If $Q = E^n$, then it is necessary to assume $f = O(r^{-2})^r$ or a similar condition.

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